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B.Sc. part - II (H), paper - III, Date - 21-05-2020
Real Analysis.

Q.1 Prove that the sequence $\{a_n\}$ defined by $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ diverges.

Soln. Here, it is given that $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n}$

Changing n to $n+1$, we get

$$a_{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{n+1}$$

$\therefore a_{n+1} - a_n = \frac{1}{n+1} > 0$, for all n

or, $a_{n+1} > a_n$.

Therefore, the sequence $\{a_n\}$ is monotonic increasing.

Now, $a_{n+p} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+p}$, $p \in N$.

$$\therefore \left| a_{n+p} - a_n \right| = \left| \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+p} \right|$$

Taking $p = n$, we get

$$|a_{2n} - a_n| = \left| \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \right|$$

or, $|a_{2n} - a_n| = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$,

as $|x| = x$, when $x > 0$.

or, $|a_{2n} - a_n| > \frac{1}{n+n} + \frac{1}{n+n} + \cdots + \frac{1}{2n}$

or, $|a_{2n} - a_n| > \frac{n}{2n} = \frac{1}{2}$, for all n. —— (I)

Let us suppose that the sequence $\{a_n\}$ is convergent. Therefore, by Cauchy's general principle of convergence, given $\epsilon > 0$, however small, there exists a positive integer m, such that

$|a_{n+p} - a_n| < \epsilon$ for $n \geq m$ and $p \in N$

Taking $p = m = n$, we get

$|a_{2n} - a_n| < \epsilon$ for all n —— II

Let ϵ be such that $0 < \epsilon < \frac{1}{2}$

Then, from I,

$|a_{2n} - a_n| > \frac{1}{2} > \epsilon$

This is a contradiction to II

Therefore, our supposition is wrong.

Hence, $\{a_n\}$ cannot converge.

We find that the sequence $\{a_n\}$ is monotone increasing and does not converge and therefore, it must diverge.

Q.2. Show that the sequence $\{u_n\}$ defined by $u_1 = \sqrt{2}$ and $u_{n+1} = \sqrt{2u_n}$ converges to 2.

Soln: We have $u_1 = \sqrt{2}$ — I And also $u_{n+1} = \sqrt{2u_n}$ —— II

putting $n=1$ in II, we get $u_2 = \sqrt{2u_1} = \sqrt{2\sqrt{2}}$, with the help of I.

Evidently $\sqrt{2} > 1$ or $2\sqrt{2} > 2$ or $\sqrt{2\sqrt{2}} > \sqrt{2}$ or $u_2 > u_1$

Let $u_{n+1} > u_n$. Then $2u_{n+1} > 2u_n$ or $\sqrt{2u_{n+1}} > \sqrt{2u_n}$

or, $u_{n+2} > u_{n+1}$, with the help of II

$\therefore$ $u_{n+1} > u_n$ implies that $u_{n+2} > u_{n+1}$

Hence, by mathematical induction, the given sequence $\{u_n\}$ is monotone increasing.

Now, $u_1 = \sqrt{2}$. Evidently $\sqrt{2} < 2$. $\therefore u_1 < 2$

Let $u_n < 2$. Then, by II, $u_{n+1} = \sqrt{2u_n} < \sqrt{2 \cdot 2} = 2$

$\therefore$ $u_n < 2$ implies that $u_{n+1} < 2$.

Hence, by mathematical induction, $u_n < 2$, for all $n \in N$.

Therefore, the sequence $\{u_n\}$ is bounded above.

Thus, it must be convergent. Let $\lim_{n\to\infty} u_n = l$

By II, $u_{n+1} = \sqrt{2u_n}$, or $u_{n+1}^2 = 2u_n$ or, $\lim_{n\to\infty} u_{n+1}^2 = 2 \lim_{n\to\infty} u_n$ or, $l^2 = 2l$

or, $l = 2$, as $l \neq 0$. $\therefore$ $\sqrt{2} = u_1 < u_2 < u_3 < \cdots\cdots$

$\therefore$ $\{u_n\}$ is bounded below by $\sqrt{2}$.

$\therefore$ $\lim_{n\to\infty} u_n \geq \sqrt{2}$

Hence, $l = 2$. Proved.